Distance in Schwarzschild spacetime

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1 Abstract

We show the radial proper distance measured by a radially-moving observer in Schwarzschild spacetime is $\frac{1}{|e|}dr$, where e is the energy per mass. This relates their ruler measurement to the Schwarzschild r-coordinate, and is valid for all $r > 0$. On the other hand, these moving observers measure static objects to occupy distance $e\Big(1 - \frac{2M}{r}\Big)^{-1} dr$, which applies for $r > 2M$. Both quantities reduce to the familiar "proper distance" $\Big(1 - \frac{2M}{r}\Big)^{-1/2} dr$ when the measuring observer is static, as expected.

We overview four different tools to measure distance: adapted coordinates, the spatial projector, tetrads, and the radar metric. Because spatial measurement has been underdeveloped in relativity, we re-derive our results using each tool, using complementary results to justify the unfamiliar. Likewise, we overview three approaches to generate new coordinate systems adapted to a given observer field: a Lorentz boost of coordinates, falling clocks, and the dual velocity. This theory is easily transferred to other frames and spacetimes.

We derive generalisations of Gullstrand-Painlevé and Lemaître coordinates, which are adapted to the moving observers. We introduce "snow" observers which may only exist inside the event horizon. We highlight a subtlety about coordinate vectors and the relativity of the "radial" direction, which is key to our results. Many explanations foster a "Newtonian" misconception of an absolute background space, but we emphasise the relativity of length.

2 Introduction

2.1 Static distance

From the line element for Schwarzschild-Droste coordinates,

$$ds^2 = -\Big(1 - \frac{2M}{r}\Big)dt^2 + \Big(1 - \frac{2M}{r}\Big)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{1}$$

the radial "proper distance" follows from setting $dt = d\theta = d\phi = 0$:

$$ds = \Big(1 - \frac{2M}{r}\Big)^{-1/2} dr \tag{2}$$

as presented in any introductory setting. However important questions tend to be left unanswered: Why does the factor $\Big(1 - \frac{2M}{r}\Big)^{-1/2}$ become imaginary for $r < 2M$? What is the radial distance inside the horizon? Why is there no "according to..." disclaimer when special relativity pedagogy is so clear length is relative to the observer? Why call this specific case the "proper distance" when the general definition $\int ds$ is over an arbitrary path?

Indeed, if we repeat the procedure in Gullstrand-Painlevé coordinates:

$$ds^2 = -\Big(1 - \frac{2M}{r}\Big) dT^2 + 2\sqrt{\frac{2M}{r}} dT\, dr + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{3}$$

now with $dT = 0$, the result is simply $1\, dr$. In Eddington-Finkelstein coordinates (null version), the corresponding quantity is zero! As we shall see, it is *static* observers who determine the $\Big(1 - \frac{2M}{r}\Big)^{-1/2} dr$ radial distance, or equivalently, it is measured along a static slice of spacetime. This link is rarely made, although one exception is Taylor & Wheeler (2000, §2.9, §2.10). While advanced textbooks rigorously justify the static slicing, which is interpreted as the most natural or preferred, they fail to link it with the $\Big(1 - \frac{2M}{r}\Big)^{-1/2} dr$ distance (e.g. Wald, 1984, §6.1, §6.3). (More generally, all stationary observers — who have fixed r but possibly angular motion — measure this radial distance.)

Recall Schwarzschild spacetime is static for $r > 2M$ because it has a Killing vector field which is timelike there, and which is orthogonal to a set of spatial hypersurfaces. The Killing vector field ξ is the unique one which is timelike for all $r > 2M$, and in Schwarzschild-Droste coordinates is $\xi^\mu = (1, 0, 0, 0)$. For $r > 2M$, the static hypersurfaces $t = const$ are spatial and orthogonal to ξ, and form a natural choice along which to measure distance. Alternatively, the Killing field defines *static observers* $\mathbf{u} \equiv \xi/\sqrt{-\xi\cdot\xi}$ which point in the same 4-direction but are normalised. They remain at fixed spatial "location" (r, θ, ϕ) and passage only through time, with 4-velocity

$$u^\mu = \left(\Big(1 - \frac{2M}{r}\Big)^{-1/2}, 0, 0, 0\right) \tag{4}$$

Dinunno & Matzner (2010)'s comment about another spatial measurement applies equally to radial distance:

> Elucidating these results for the volume provides a new pedagogical resource of facts already known in principle to the relativity community, but rarely worked out.

2.2 Rain, hail, drip, snow

Since distance is dependent on motion, we must clarify which motions are possible. For timelike radial geodesics, Taylor & Wheeler (2000, §B.2) introduced the metaphors "rain", "hail", and "drips", based on the Earthly parallels. Rain observers fall from rest at infinity, loosely speaking. Hail observers fall faster than rain, effectively starting with an initial inward push at infinity. Drips fall from rest at finite $r_i > 2M$. The authors claim these "cover all possible radially moving free-float frames" (§B.2), however there are others which we dub "snow". These are achieved by passing inside the horizon then accelerating sufficiently "outwards".

Name	Energy per mass	Traditional name	Allowed locations
hail	$e > 1$	hyperbolic	all
rain	$e = 1$	parabolic	all
drip	$0 < e < 1$	elliptic	$r \le r_i = \frac{2M}{1-e^2}$
snow	$e \le 0$	—	$r < 2M$

The Killing field ξ gives rise to a quantity:

$$e \equiv -\mathbf{u}\cdot\xi \tag{5}$$

interpreted as the "total energy per unit mass as measured at infinity". Because it is conserved along geodesics, it gives a natural and elegant parametrisation of freely-falling motion. (Another popular parameter choice is r_i, however this covers only the "drip" cases.) For $r > 2M$, a given e corresponds to two radial motions, one ingoing and one outgoing: we assume only ingoing motion whereupon e is unique. We term rain/hail/drip/snow "moving" or "falling" observers. Their 4-velocity is

$$u^\mu = \left(e\Big(1 - \frac{2M}{r}\Big)^{-1}, -\sqrt{e^2 - 1 + \frac{2M}{r}}, 0, 0\right) \tag{6}$$

which follows from radial motion, Equation 5, and normalisation. Assume components are given in Schwarzschild-Droste coordinates throughout, unless otherwise specified. Static observers have $e = \sqrt{1 - \frac{2M}{r}}$, note the energy per mass is still well-defined for non-inertial motion.

2.3 Moving distance

There is very little literature on the distance measured by the radial fallers, and we scramble to assemble even peripherally-related results. The definitive but underappreciated paper is Gautreau & Hoffmann (1978). For the drip case ($0 < e < 1$), they give the proper distance as (their Equation 13)

$$L = \int ds = \frac{r_2 - r_1}{\left(1 - \frac{2M}{r_i}\right)^{1/2}} \tag{7}$$

based on "drip" coordinates. In our notation this is $dL = \frac{1}{e} dr$ since $e = \sqrt{1 - \frac{2M}{r_i}}$ is determined from the initial $r = r_i$. Building on their work, Finch (2015, §2, §3) gives the 3-volume inside the horizon as $\frac{1}{e}\frac{4\pi(2M)^3}{3}$ which is also consistent with our result.

In the rain case ($e = 1$), Taylor & Wheeler (2000, §B.3) show the radial distance is simply dr. Lemaître (1932, §11) and later authors point out the 3-space of Gullstrand-Painlevé coordinates is flat[1] although do not articulate the consequence for r.

3 Generalised Gullstrand-Painlevé and Lemaître coordinates

A major aspect of our investigation is the use of different coordinate systems to give different "perspectives". Of course measurables must be independent of coordinates, however suitably chosen coordinates may present measurables more clearly. Also the use of complementary alternatives draws out an easily-missed subtlety about the direction "radial", and clarifies which quantity one is actually measuring!

We present a family of coordinates parametrised by energy per mass $-\infty < e < \infty$, $e \neq 0$:

$$ds^2 = -\frac{1}{e^2}\left(1 - \frac{2M}{r}\right)dT^2 + \frac{2}{e^2}\sqrt{e^2 - 1 + \frac{2M}{r}}\,dT\,dr + \frac{1}{e^2}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{8}$$

where r, θ, and ϕ are the usual Schwarzschild coordinates, but $T = T_e$ is defined by:

$$dT = e\,dt + \left(1 - \frac{2M}{r}\right)^{-1}\sqrt{e^2 - 1 + \frac{2M}{r}}\,dr \tag{9}$$

T is the proper time for falling observers of the same e (Section 4.2), with whom we assume spacetime is filled. For $e < 1$ the coordinates do not cover all $r > 0$, see Table 2.2. [2] These coordinates are "ingoing", but one can easily define an "outgoing" variant by changing the first "+" sign in each equation to a "−". They can also be extended into the "white hole" analytic continuation of spacetime (c.f. Kraus & Wilczek, 1994, §2)[3]. To show the function T exists, the above equation must match the total differential $dT = \frac{\partial T}{\partial t}dt + \frac{\partial T}{\partial r}dr$, which is clear since the coefficient of dr is a function of r only. The transformation does fail at $r = 2M$ as expected. See online for closed form solutions[4].

The $e = 1$ case was discovered by Gullstrand (1922) and Painlevé (1921), and rediscovered by many others from Lemaître (1932) onwards[5]. Gautreau & Hoffmann (1978) derived the "drip" case $0 < e < 1$. Martel & Poisson (2001) added the"hail" case, although with a slightly different time coordinate for which the coordinate family

[1] Check: Lemaitre wrote "At each instant, space is Euclidean", but I need to examine again
[2] to do: add diagram here
[3] to do: check this
[4] To do...
[5] check that Lemaitre re-discovered

contains Eddington-Finkelstein coordinates as the limiting case $e \to \infty$ of photons (§3). Finch (2015) rescaled the hail coordinate to be proper time, unifying the coordinates for all $e > 0$; see also other contributors cited in these papers. I have extended the coordinates to the "snow" case $e < 0$, changed the parameter to e, and added the "outgoing" variant. While the line element is formally the same for e and $-e$, the meaning of T is different.

Another coordinate family is also helpful, in which the metric takes form

$$ds^2 = -dT^2 + \frac{1}{e^2}(e^2 - 1 + \frac{2M}{r})d\rho^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{10}$$

T is the same as above, but $\rho = \rho_e$ is defined by

$$d\rho = e\, dt + e^2\Big(1 - \frac{2M}{r}\Big)^{-1}\sqrt{e^2 - 1 + \frac{2M}{r}}dr \tag{11}$$

The $e = 1$ case is Lemaître coordinates (Lemaître, 1932, §11). Gautreau & Hoffmann (1978, Equation 22) have a similar expression for the $0 < e < 1$ case[6]. I obtained this generalisation by requiring a new coordinate $\rho = \frac{\partial\rho}{\partial t}dt + \frac{\partial\rho}{\partial r}dr$ to yield a diagonal metric. These coordinates are even more closely adapted to the e-fallers, since additionally they are comoving and the metric is diagonal, although ρ is less familiar. As before $-\infty < e < \infty$, $e \neq 0$.

One class was omitted, the "zero energy" $e = 0$ observers. These are better known by their property of maximising the proper time inside the horizon. Schwarzschild-Droste coordinates are best suited, because in them the observers are comoving ($t = const$). For static observers, Schwarzschild-Droste are also the best suited, because they are comoving ($r = const$).

4 Three techniques to generate coordinates

We review three methods to form the generalised Gullstrand-Painlevé coordinates given above. This is intended as a resource with wider applicability.

4.1 Lorentz boost

One technique to derive new coodinates is to perform a Lorentz boost on the coordinates themselves, from Schwarzschild-Droste to generalised Gullstrand-Painlevé. What we really mean is a boost from the static frames to falling frames. We generalise Taylor & Wheeler (2000, §B.4) (attributed to Misner, see §B.11) whose derivation for the $e = 1$ case is beautiful but contains several errors.

In curved spacetime, Lorentz boosts are allowed *locally* in an orthonormal or "Minkowski" frame, so we start with such a frame. For $r > 2M$, Schwarzschild-Droste coordinates lead naturally to static frames, as these are a simple normalisation of the coordinate vectors. Hence it is tempting to define local inertial coordinates $dt' \equiv \left(1 - \frac{2M}{r}\right)^{1/2} dt$ etc. so that the metric would have Minkowski form $ds^2 = -dt'^2 + dr'^2 + \cdots$. However no such function t' exists, besides if the metric did take this form on some neighbourhood the spacetime would be flat there. On the other hand multiplying each coordinate by a constant scalar such as $\left(1 - \frac{2M}{r_0}\right)^{1/2}$ (Taylor & Wheeler, 2nd edn, draft as of 2017)[7] at all points is mathematically consistent but does not yield the desired coordinates for $r \neq r_0$. Instead, we require a non-coordinate basis (c.f. Lin & Soo, 2013). $\left(1 - \frac{2M}{r}\right)^{1/2} dt$ is a perfectly legitimate 1-form, it just does not arise as the differential of anything. We use the labels $\mathbf{e}^{\hat{t}} \equiv \left(1 - \frac{2M}{r}\right)^{1/2} dt$ in place of dt' and $\mathbf{e}^{\hat{r}} \equiv \left(1 - \frac{2M}{r}\right)^{-1/2} dr$, because in fact these form an orthonormal frame (a dual frame, contrast this with the frame in Section 5.3), so the metric is canonical in this frame.

[6] To do: check; I find their variables unclear
[7] to do: check again

We ignore θ and ϕ which are orthogonal to the boost direction; alternatively one could normalise then revert them after the boost.

Recall in special relativity the Lorentz boost formula for time is $\overline{T} = \gamma(\bar{t} - Vx)$ between Minkowski coordinates. In place of the dummy variables t and x we substitute the time and space covectors $\mathbf{e}^{\hat{t}}$ and $\mathbf{e}^{\hat{r}}$, so the right hand side becomes

$$\gamma\left(\left(1-\frac{2M}{r}\right)^{1/2}dt + V\left(1-\frac{2M}{r}\right)^{-1/2}dr\right) \tag{12}$$

The Lorentz factor is $\gamma = -\mathbf{u}_{\text{static}} \cdot \mathbf{u}_{\text{faller}} = e\left(1-\frac{2M}{r}\right)^{-1/2}$, using Equations 4 and 6. The 3-speed is hence $V = -\sqrt{1-\gamma^{-2}} = -\frac{1}{e}\sqrt{e^2 - 1 + \frac{2M}{r}}$, taking the sign to be consistent with the r-axis. Substituting yields Equation 9 which we showed arises from a coordinate, so the boosted frame is holonomic! (We used the Lorentz boost and not its inverse, since the coordinate dual basis transforms oppositely to the coordinate basis, and the coordinate basis transforms oppositely to vector components (Schutz, 2009, §3.3).)

Curiously, this procedure gives the intended coordinates even for $r \leq 2M$ where static observers are not physical, and the "Lorentz boost" is faster than light, with interim quantities γ and $\left(1-\frac{2M}{r}\right)^{\pm 1/2}$ complex numbers ($r < 2M$). Taylor & Wheeler (2000) use the result without mention. Misner et al. (1973, §31.2) boost the Riemann tensor between these same frames in the $e = 1$ case, and explicitly acknowledge $|V| > 1$ but without justification. In the very least, for $r \leq 2M$ we can treat the derivation as merely heuristic, with the rigour simply that: (a) the coordinate transformation is a diffeomorphism (or directly, the metric satisfies the field equations), and (b) T is proper time, as shown next[8].

4.2 Dual velocity

Martel & Poisson (2001) give a simple and elegant procedure using the dual velocity of the observer field, which they applied to derive "hail" coordinates. The dual velocity, which has components $u_\mu = g_{\mu\nu}u^\nu$, is most suggestive when expressed in terms of the dual basis vectors:

$$\mathbf{u} = -e\,\mathbf{dt} - \left(1-\frac{2M}{r}\right)^{-1}\sqrt{e^2 - 1 + \frac{2M}{r}}\,\mathbf{dr} \tag{13}$$

This looks like a total differential $\frac{\partial\cdot}{\partial t}dt + \frac{\partial\cdot}{\partial r}dr$. Hence we seek a new coordinate T satisfying $\mathbf{dT} \equiv -\mathbf{u}$, that is $\frac{\partial T}{\partial x^\mu} = -u_\mu$. (The minus sign is merely due to our metric signature $-+++$. Martel & Poisson (2001) use coefficient $-e$ because their time coordinate is scaled differently.) T exists as shown before, or alternatively in the language of differential forms, dT is closed: $d(\mathbf{dT}) = 0$, hence it is exact: $d(T)$.

That T is proper time follows from the total derivative

$$\frac{dT}{d\tau} = \frac{\partial T}{\partial x^\mu}\frac{dx^\mu}{d\tau} = -u_\mu u^\mu = 1 \tag{14}$$

According to Martel & Poisson (2001, §5) the dual velocity procedure can be generalised to other static, spherically symmetric spacetimes, although a scalar "integrating factor" may be needed.

4.3 Dropped clocks

A more "physically" motivated construction is to map spacetime using the proper time of freely falling clocks. Gautreau & Hoffmann (1978, §3) conceived a "clock factory" at fixed r_i which drops clocks at regular intervals, mapping out the "drip" ($0 < e < 1$) coordinates by their proper time. Moore (2012) gives an extended pedagogical treatment for rain ($e = 1$) by considering $\frac{\partial T}{\partial t}$ and $\frac{\partial T}{\partial r}$ physically, which we extend to the hail case.

[8] Check: maybe we can boost from $e = 0$ Schwarzschild-Droste frame instead? This would still omit $r = 2M$ but we could argue from continuity.

(Apparently Novikov earlier considered the time on falling objects, but with a different setup and resulting coordinates (Novikov, 1963) as cited in (Misner et al., 1973, §31.4).)

Consider a static clock factory at spatial infinity, so its "master clock" reads Schwarzschild t. The hail case is complicated by the initial relative motion, which has Lorentz factor $\gamma = e$. The clocks take their initial time from the factory, but because of time-dilation, at infinity $\frac{dT}{dt} = e$.[9]

To find $\frac{\partial T}{\partial t}$ physically, consider two events at some fixed r but separated by dt. These correspond to the worldlines of two different clocks, which are identical apart from translation in t, hence they had the same separation dt at infinity also. This means a T difference of $e\,dt$, hence $\frac{\partial T}{\partial t} = e$.

Now to find $\frac{\partial T}{\partial r}$, we could take a total derivative like $\frac{dT}{dr} = \frac{\partial T}{\partial t}\frac{dt}{dr} + \frac{\partial T}{\partial r}$ and solve for $\frac{\partial T}{\partial r}$, which is possibly how Gautreau & Hoffmann (1978) derived their Equation 8 in the drip case. But it is enlightening to derive this "physically": $\frac{\partial T}{\partial r}$ is determined from two events at the same t but separated by dr. There are two competing effects: the lower clock was dropped earlier so had a lesser starting value of T, however it has also fallen for longer. Falling by an extra dr takes a time $dT' = \frac{dr}{dr/dT} = \frac{dr}{u^r} = \left(e^2 - 1 + \frac{2M}{r}\right)^{-1/2} dr$. The lower clock must have been released earlier by a coordinate interval $dt = \frac{dr}{dr/dt} = \frac{dr}{u^r/u^t} = -e\left(1 - \frac{2M}{r}\right)^{-1}\left(e^2 - 1 + \frac{2M}{r}\right)^{-1/2} dr$, but as before the clock time dT'' is e times this. Combining these effects gives a total $dT = dT' + dT'' = -\left(1 - \frac{2M}{r}\right)^{-1}\sqrt{e^2 - 1 + \frac{2M}{r}}\,dr$, and after restoring freedom to both coordinates we get the total differential as before.

For the "snow" case $e \leq 0$, the clocks likewise cannot be dropped from rest. One scenario would be to let the factories themselves fall freely until reaching some predetermined $r = 2M - \epsilon$ when they blast a clock "upwards", after compensating appropriately for time-dilation.

5 Four tools to measure distance

5.1 Direct coordinate measurement

Given the line element in Equation 10 it is trivial to set $dT = d\theta = d\phi = 0$ to obtain

$$ds = \frac{1}{|e|} dr \tag{15}$$

Note this is along a slice of constant faller proper time. More rigourously, this uses the general definition of *proper distance* $\int ds$, that is, $\int \sqrt{g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\,d\lambda$ over a spacelike path parameterised by λ (these equations look more familiar as integrals). Just take $\lambda = r$.[10] This procedure has limited flexibility, but is simple and pedagogical.[11]

We use the term "adapted" coordinates only loosely. In an arbitrary spacetime, suppose for coordinates x^μ an observer u^μ wishes to measure in the x^1-direction. Then the distance is $g_{11}^{1/2} dx^1$ only when $u_1 = 0$; for a diagonal metric this is equivalent to $u^1 = 0$ so the observer is at constant x^1. Alternatively if the observer is comoving, it suffices that $g_{01} = 0$. These results are straightforward using the spatial projector, but as we will see one must be very careful with coordinate directions.

5.2 Spatial projector

The *spatial projection tensor* $\mathbf{P}$ projects tensor quantities orthogonally onto the spatial part of an observer's frame. Cattàneo (1958, and subsequent) provided the first in-depth analysis of the projector. For a given

[9] To do: justify why we don't interpret time-dilation the reverse way.

[10] Check: I assume this parameter does not need to be affine?

[11] Diagram? Explain this is slicing-dependent, show graphs of slicing. G-P slices in Schw coords, or vice versa.

observer $\mathbf{u}$,

$$P_{\mu\nu} \equiv g_{\mu\nu} + u_\mu u_\nu \tag{16}$$

It is also termed the *spatial metric* because it gives spatial measurements in the observer's frame by $dL^2 = P_{\mu\nu} dx^\mu dx^\nu$ (incidentally this expression is just $\mathbf{P}$ in a coordinate basis).

For Schwarzschild-Droste coordinates and an e-faller from Equation 13,

$$P_{\mu\nu} = \begin{pmatrix} e^2 - 1 + \frac{2M}{r} & e\sqrt{e^2 - 1 + \frac{2M}{r}}\left(1 - \frac{2M}{r}\right)^{-1} & 0 & 0 \\ e\sqrt{e^2 - 1 + \frac{2M}{r}}\left(1 - \frac{2M}{r}\right)^{-1} & \boxed{e^2\left(1 - \frac{2M}{r}\right)^{-2}} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \tag{17}$$

We are only concerned with the boxed component P_{rr}, which yields

$$dL = \left| e\left(1 - \frac{2M}{r}\right)^{-1} \right| dr \tag{18}$$

since distance is defined to be positive. However there is an apparent discrepancy with the rain/hail/drip/snow coordinates which adds much insight:

$$P_{\mu\nu} = \begin{pmatrix} \frac{1}{e^2}(e^2 - 1 + \frac{2M}{r}) & \frac{1}{e^2}\sqrt{e^2 - 1 + \frac{2M}{r}} & 0 & 0 \\ \frac{1}{e^2}\sqrt{e^2 - 1 + \frac{2M}{r}} & \boxed{\frac{1}{e^2}} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \tag{19}$$

In this case, the radial distance is simply

$$dL = \frac{1}{|e|} dr \tag{20}$$

Intuitively, "distance is inversely proportional to energy". This latter result concurs with (Gautreau & Hoffmann, 1978, Equation 13) who consider only the drip case $0 < e < 1$. The results from both coordinates reduce to the familiar quantity $\left(1 - \frac{2M}{r}\right)^{-1/2} dr$ for static observers.

But why the difference? It is no fault of $\mathbf{P} = \mathbf{g} + \mathbf{u} \otimes \mathbf{u}$ which is a tensor, so the above matrices represent the same object. It turns out it is the radial vectors which are distinct, for example $(\partial_r)^\mu = (0, 1, 0, 0)$ in rain/hail/drip/snow coordinates transforms to

$$\left(-\frac{1}{e}\left(1 - \frac{2M}{r}\right)^{-1}\sqrt{e^2 - 1 + \frac{2M}{r}}, 1, 0, 0\right) \tag{21}$$

in Schwarzschild-Droste coordinates, which is clearly not the Schwarzschild-Droste r-coordinate vector $(0, 1, 0, 0)$. Hence we label them ${}_{\text{(faller)}}\partial_r$ and ${}_{\text{(static)}}\partial_r$ respectively[12]. The unexpected fact is that while the r-coordinates themselves are identical (in the sense of functions $r : \mathcal{M}^4 \to \mathbb{R}$ on the manifold), the coordinate *vectors* are influenced by the other coordinates bundled together with them.

$\mathbf{P}$ acts on two vectors $\mathbf{a}$ and $\mathbf{b}$ say, by contraction: $\mathbf{P}(\mathbf{a}, \mathbf{b}) = P_{\mu\nu} a^\mu b^\nu$. So we can write, in index-free notation:

$$\mathbf{P}({}_{\text{(static)}}\partial_{\mathbf{r}}, {}_{\text{(static)}}\partial_{\mathbf{r}}) = e^2\left(1 - \frac{2M}{r}\right)^{-2} \tag{22}$$

$$\mathbf{P}({}_{\text{(faller)}}\partial_{\mathbf{r}}, {}_{\text{(faller)}}\partial_{\mathbf{r}}) = \frac{1}{e^2} \tag{23}$$

These results are the same when evaluated in either coordinate system, as expected. Taking the square root gives the spatial length of each vector, which concurs with the previous results.

[12]Think: should I rename coordinate vectors to "Schw" and "GP"?

The physical interpretation of the coordinate vector difference is that each observer determines "radial" differently. What one observer determines to be a purely spatial direction, the other determines to mix up time with space. Tetrad frames quantify this precisely.

5.3 Tetrads

A *tetrad* is a set of four vector fields which form a basis for the tangent space at each point. They were introduced by Weyl, Einstein, and Wigner in 1928–1929. The frame of a timelike observer $\mathbf{u}$ may be specified by an orthonormal tetrad $(\mathbf{e}_{\hat{\alpha}})$, where $\mathbf{e}_{\hat{0}} \equiv \mathbf{u}$ is the observer's "time" direction and $(\mathbf{e}_{\hat{1}}, \mathbf{e}_{\hat{2}}, \mathbf{e}_{\hat{3}})$ span the observer's 3-space. While the general procedure of decomposing tensors in an orthonormal frame to extract measureable quantities is well known, I have not seen this applied to distance measurement specifically.

The natural frame for static observers ($r > 2M$) is a simple normalisation of the Schwarzschild-Droste coordinate vectors:

$$
\begin{aligned}
{}_{(\text{static})}e^{\mu}_{\hat{0}} &= \left(\left(1-\frac{2M}{r}\right)^{-1/2}, 0, 0, 0\right) \\
{}_{(\text{static})}e^{\mu}_{\hat{1}} &= \left(0, \left(1-\frac{2M}{r}\right)^{1/2}, 0, 0\right) \\
{}_{(\text{static})}e^{\mu}_{\hat{2}} &= \left(0, 0, \frac{1}{r}, 0\right) \\
{}_{(\text{static})}e^{\mu}_{\hat{3}} &= \left(0, 0, 0, \frac{1}{r\sin\theta}\right)
\end{aligned}
$$

The natural e-faller tetrad is:

$$
\begin{aligned}
{}_{(\text{faller})}e^{\mu}_{\hat{0}} &= \left(e\left(1-\frac{2M}{r}\right)^{-1}, \sqrt{e^2-1+\frac{2M}{r}}, 0, 0\right) \\
{}_{(\text{faller})}e^{\mu}_{\hat{1}} &= \left(\left(1-\frac{2M}{r}\right)^{-1}\sqrt{e^2-1+\frac{2M}{r}}, e, 0, 0\right)
\end{aligned}
$$

where $\mathbf{e}_{\hat{2}}$ and $\mathbf{e}_{\hat{3}}$ are the same as for the static observer. These orthonormal tetrads are unique, given the orientation of the coordinates θ, ϕ and r, and a requirement that the $\mathbf{e}_{\hat{1}}$ be "radial" (meaning orthogonal to θ and ϕ)[13]. Note the $\mathbf{e}_{\hat{1}}$ are the coordinate vectors ${}_{(\text{static})}\partial_{\mathbf{r}}$ and ${}_{(\text{faller})}\partial_{\mathbf{r}}$ (Equation 21) but normalised. The faller's radial vector appears to intrude into "time", but this is just an artefact of the coordinate choice. To get the observers' measurements, we express the coordinate vectors ∂_r in each frame.

Recall the decomposition of a vector $\mathbf{a}$ into an orthonormal basis $(\mathbf{e}_{\hat{\mu}})$ is

$$
\mathbf{a} = (\mathbf{a}\cdot\mathbf{e}_{\hat{0}})\mathbf{e}_{\hat{0}} + (\mathbf{a}\cdot\mathbf{e}_{\hat{1}})\mathbf{e}_{\hat{1}} \tag{24}
$$

Hence ${}_{(\text{static})}\partial_{\mathbf{r}}$ decomposes into each frame as:[14]

$$
{}_{(\text{static})}\partial_r = \left(1-\frac{2M}{r}\right)^{-1/2}_{\textbf{(static)}}\mathbf{e}_{\hat{1}} \tag{25}
$$

$$
{}_{(\text{static})}\partial_r = -\left(1-\frac{2M}{r}\right)^{-1}\sqrt{e^2-1+\frac{2M}{r}}_{\,(\text{faller})}\mathbf{e}_{\hat{0}} + e\left(1-\frac{2M}{r}\right)^{-1}_{(\text{faller})}\mathbf{e}_{\hat{1}} \tag{26}
$$

As in the last section, this index-free notation highlights the geometric meaning without the distraction of coordinates. The first equation says that to the static observer (right hand side), ${}_{(\text{static})}\partial_r$ is purely spatial — indeed radial — and covers the familiar distance $\left(1-\frac{2M}{r}\right)^{-1/2}$. The second equation says that to the

[13] check: in a rotating coordinate system, would this still be the definition of radial??
[14] Diagram: show tetrads and decomposition in 1+1-dimensions

falling observer, the same vector $_{(\text{static})}\partial_r$ is a mix of time and space; we ignore the "time" component (as is done for length-contraction in special relativity, or in the spatial metric for instance), hence the spatial part has length $e\left(1-\frac{2M}{r}\right)^{-1}$ as measured in this frame. Note the $\mathbf{e}_{\hat{1}}$ are essentially idealised rigid rulers of unit length, although technically they reside in the tangent space not along the manifold.[15]

The other coordinate vector decomposes as:

$$_{(\text{faller})}\partial_r = \frac{1}{e}\,_{(\text{faller})}\mathbf{e}_{\hat{1}}$$

$$_{(\text{faller})}\partial_r = \frac{1}{e}\left(1-\frac{2M}{r}\right)^{-1/2}\sqrt{e^2-1+\frac{2M}{r}}\,_{(\text{static})}\mathbf{e}_{\hat{0}} + \left(1-\frac{2M}{r}\right)^{-1/2}_{(\text{static})}\mathbf{e}_{\hat{1}}$$

This means that in the faller's frame, the faller radial coordinate vector ∂_r is purely spatial, with proper distance $\frac{1}{e}$. However in the static frame, this is a displacement through time as well as a proper distance $\left(1-\frac{2M}{r}\right)^{-1/2}$ in the radial direction.

5.4 Radar metric

In relativity, "*radar*" *distance* is determined by the travel time for photons to reflect off an object and return to the source. The concept was promoted by Milne in the 1930s, Landau & Lifshitz (1941, §79), Bondi, and others. The concept rose to prominence as an alternative to the rigid ruler, a construct which the early relativists moved away from. Since c is independent of the observer, radar may seem a preferred measurement of "actual distances and time intervals", or to represent the "geometric properties" or "metric of real space" (Landau & Lifshitz, 1994, §84). Hence some conclude "in relativity, time is a primary notion and length a derived one" (Gourgoulhon, 2013, §3.3.1), however one unorthodox feature of this interpretation is that the constancy of c reduces to mere convention (Bondi, 1964, §4), and besides there is no issue with *somewhat* rigid rulers (Rindler, 1977, §2.5). Some authors are critical of the physical significance at macroscopic distances (Bini et al., 2005).

The radar distance is $frac12\tau$, where τ is the proper time of the observer holding the radar gun. Locally, $ds^2 = \gamma_{ij}dx^i dx^j$, where γ is the Landau-Lifshitz radar metric:

$$\gamma_{ij} \equiv g_{ij} - \frac{g_{0i}g_{0j}}{g_{00}} \tag{27}$$

where $i, j = 1, 2, 3$, so γ is restricted to the 3-dimensional subspace spanned by the coordinate vectors ∂_i. Note the formula assumes the observers are *comoving* with the given coordinates. Hence to apply this formula for the fallers, we need the generalised Lemaître coordinates, in which:

$$\gamma_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{e^2}\left(e^2-1+\frac{2M}{r}\right) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \tag{28}$$

We have reinterpreted γ as a 4-dimensional tensor to aid transformation, the $\gamma_{0\alpha}$ terms are normally omitted because they are always zero. The γ_{11} component relates distance in the ρ-direction, however we want distance in terms of r. We can either transform the radar metric to our previous coordinate systems, which yields exactly the spatial metric (Equations 17 and 19), or transform the radial vectors $_{(\text{static})}\partial_{\mathbf{r}}$ and $_{(\text{faller})}\partial_{\mathbf{r}}$ into these generalised Lemaître coordinates and evaluate. This concurs with the previous results, where in this case we say the distance depends on which observer is holding the radar gun, via their proper time.

[15] To do: edit or delete this further explanation: more precisely this is in the tangent space, and is a ratio of distance to coordinate. The modern interpretation of differentials is as a ratio, that is $dL = \left(1-\frac{2M}{r}\right)^{-1/2}dr$. We might think of this as an "infinitesimal" interval on the manifold itself (between different points). ∂_r is a finite vector, but this is only in the tangent space at a single point, not across the manifold... Also, why use the coordinate vector, and not $\mathbf{e}_{\hat{1}}$ for instance? This is for convenience, because the coordinate vectors have component $u^r = 1$. So what we are asking is, when r changes by 1, how does the distance change?

It is straightforward to show the equivalence of radar distance and spatial projector distance generally (c.f. Bini et al., 2005, Appendix B). Given a spacetime and a 4-velocity $\mathbf{u}$, find coordinates for which $\mathbf{u}$ is comoving[16]. Then $\mathbf{u} \cdot \mathbf{u} = -1$ implies $g_{00} < 0$, so the 4-velocity has components $u^\mu = \big((-g_{00})^{-1/2}, 0, 0, 0\big)$, assuming the x^0-coordinate is forward-pointing. It follows that the spatial projector is $P_{\mu\nu} = g_{\mu\nu} - \frac{g_{0\mu}g_{0\nu}}{g_{00}}$, but these are the same components as the radar metric (4-dimensional version)! Since the components are the same in this coordinate system, the objects are identical, since we expect the radar metric to transform as a tensor because it corresponds to observables. Radar is also defined for reflection events at large distances, but for *local* measurement it is the same as the spatial metric.

6 Discussion

The $\left(1-\frac{2M}{r}\right)^{-1/2} dr$ radial distance should be qualified "...according to static or stationary observers". There is a danger that the Euclidean fixed background of pre-relativistic physics is merely replaced conceptually by the "funnel"-shaped fixed background. Eisenstaedt (1989) contends the relativity community had an essentially "neo-Newtonian" interpretation of general relativity — particularly of Schwarzschild spacetime — until the 1960s, however such misconceptions remain today. Constructively, if applying a $3 + 1$ approach like the ADM formalism or quantum fields on curved spacetime, it helps to be clear about the choice of background slicing.

The static frames are a natural choice because they best respect the spherical and time symmetries of the spacetime. However I suggest the rain frame congruences are just as deserving, because freely-falling frames are especially natural in relativity (Taylor & Wheeler, 1992, §2.2), and rain observers are unique in having perfectly balanced energy (zero kinetic energy at infinity). Also in terms of physical practicality, these frames are the most realistically attainable by a human-made rocket for instance.

Historically, coordinates suited to timelike observers have been underappreciated in favour of coordinates suited to photons (Finch, 2015, §1). Had Gullstrand-Painlevé been as popular as Schwarzschild-Droste, the richness of their contrasting perspectives — in which the radial distance is clear from simple inspection of the line elements — might have been better appreciated. Ironically, many modern textbooks categorically state dr (or r) is *not* the distance but that $\left(1-\frac{2M}{r}\right)^{-1/2} dr$ is! Instead we should say that in relativity, the relation of coordinates to proper distance depends on the observer.

One objection is that r is just a coordinate. Indeed, measurables are independent of coordinates, but we want some bookkeeping reference system, and the "reduced circumference" is an especially convenient choice. Likewise latitude and longitude on Earth are arbitrary coordinates, but convention found them useful for relating position and hence also distance. We have avoided many other definitions and issues by sticking with local measurements, meaning infinitesimally close. We examine many concepts further in a forthcoming series.

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[16]Landau & Lifshitz seem to give conditions on when this is allowed, but I would assume it is always possible??? I guess for a congruence it may not be in general, but for a single observer, surely always possible... Also in Bini et al 2005 Appendix B

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